DEVISING PSEUDOGAMMA FUNCTIONS WITH MATHEMATICA

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Dedicated to Edwards A. Azoff on the occasion of his 65th birthday

Abstract.

A family of pseudogamma functions of a complex variable s was introduced Y. Cheng, C. B. Pomerance, G. J. Fox, and S. W. Graham, in [18], for which, we give the computations in this article.

1. INTRODUCTION

The Riemann zeta function, denoted by $\zeta(s)$, is a meromorphic complex- valued function of the complex variable, customarily written as $s = \sigma + it$ C ∈s uch that σ σ ∈ ∈R, t ∈ R, which is analytic everywhere except for s = 1, where it has a simple pole with the residue 1. For σ > 1, we have

C such that $\sigma \in$ R, $t \in$ R, which is analytic everywhere except for $s = 1$, where it has a simple pole with the residue 1. For $\sigma > 1$, we have

$$(1.1) \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - \frac{1}{p^s}}$$

We observe that this, together with the analyticity properties of (s) which follow from its definition (1.5) with the explicit definition of W1, W2, K, and q in (1.10) and the fact that ξ(s) is an entire function prove the statementthat B(s) is analytic on |s − u| = R, 1 ≤ u ≤ 2. Recalling the upperbound of ξ(s) in (1.11) from Lemma 2 and noting that, by our choice of Ω,47.545 = 1 < 1.0132, we have completed the proof of Lemma 6.0.98695ω Ω0.98695 In order to devise a family of pseudogamma functions, we shall start

with a strictly monotonic increasing sequence. We instead use s 1/2 in place of s = Reiφ from now on.Date: Drafted on July 21, 2018. Final version on May 9, 2020. The 2nd International Conference on Software Security, July 25–26, 2020, Bangalore, India, and

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International Journal of Information Technology.Ethical Statement: This article is an original work. It has been submitted only to the Journal of Analysis for publication. All authors are solemnly holding ethical standard by the journal concerning this work.

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2010 Mathematics Subject Classification. 30D99, 11M26, 11N05, 11Y35, 11A41, 11R42, 11Y40.Key words and phrases. The Riemann zeta function, pseudogamma functions, the Riemann xi-function, Wolfram Mathematica.

*The corresponding author would like to thank Noam D. Elkies, Brendan E. Hassett, Bjorn M. Poonen, Benedict H. Gross, Roland L. Graham, George E. Andrews, AndrewM. Odlyzko, and Carl B. Pomerance, for their comments and/or encouragements during the preparation of this article.Let N N and specify values φ0 := 0 < φ1 < φ2 < . . . < φN := π . Correspondingly, we let rj R+ be such that rj > rj+1 for j = 0, 1, . . ., N 1, with values to be determined later. Let R R0 > 0 be designated as above and take parameters

$$(1.2) \qquad W > V > R + \frac{7}{2}$$

In (1.3) below, we shall further restrict the V and W . For our approach in this article, we are going to resort to the ratio Nφ,V (s) , which is the ratio ofφ,Wtwo such normalized functions with W > V > R+ 7 on the circle |s− 1 | = R.

22In conformity with (1.2), we shall let

$$(1.3) \qquad V = bR + \frac{1}{2}, \quad W = cR + \frac{1}{2}, \qquad \text{with } c > b > 1 + 10^{-12},$$

from now on.

Definition 1.

We let V and W satisfy (1.2) and define N + 1dimensional vectors φ̇ and ṙ, for N ∈ N, as

$$(1.4) \qquad \dot{\varphi} := (\varphi_0, \varphi_1, \ldots, \varphi_N), \quad \dot{r} := (r_0, r_1, \ldots, r_N),$$

where φ_j and r_j, for $j = 0, 1, \ldots, N$, are restricted as above. Let us consider the family $F(R, \dot{\varphi}, V, W, \dot{r}; s)$ of functions $\nabla(s)$ with values in $\mathbb{C}$, given by

$$(1.5) \qquad \nabla(s) := \prod_{m=0 \text{ and } N} \mathbf{P}(\varphi_m, V, W, r_m; s) \prod_{n=1}^{N-1} \mathbf{Q}(\varphi_n, V, W, r_n; s),$$

where

$$(1.6) \qquad \mathbf{P}(\varphi_m, V, W, r_m; s) := \frac{(W - 1/2)^{r_m}[(s - 1/2)^2 - e^{2i\varphi_m}(V - 1/2)^2]^{r_m/2}}{(V - 1/2)^{r_m}[(s - 1/2)^2 - e^{2i\varphi_m}(W - 1/2)^2]^{r_m/2}},$$

for $m = 0$ and N, and

$$(1.7) \qquad \mathbf{Q}(\varphi_n, V, W, r_n; s) := \frac{(W-1/2)^{r_n}[(s-1/2)^4 - 2(V-1/2)^2(s-1/2)^2\cos(2\varphi_n) + (V-1/2)^4]^{r_n/4}}{(V-1/2)^{r_n}[(s-1/2)^4 - 2(W-1/2)^2(s-1/2)^2\cos(2\varphi_n) + (W-1/2)^4]^{r_n/4}}$$

for n = 1, 2, …, N − 1. We refer to the function $\nabla(s)$ in $F(R, \dot{\varphi}, V, W, \dot{r}; s)$

as a pseudogamma function. In this paper, we shall provide further estimates useful for the application in [9]; we need the following lemma concerning the two related functions

$$(1.8) \qquad \mathbf{B}(s) = \frac{\xi(s)}{\nabla(s)}, \quad \mathbf{C}(s) = \frac{\nabla(2 - X + s)}{\nabla(s)},$$

where $\frac{1}{2} < X < 1$ and $\nabla(s)$ is defined in (1.5) above (or (2.3) as in [18]).

Lemma 6. ***Let $R \geq R_0$ with R_0 defined at the end of Section 1. Suppose* that there are no zeros for the Riemann *$\xi(s)$ on the circle $|s - u| = R$, with***
$\frac{1}{2} < u \leq 2$. Then the function $\boldsymbol{B(s)}$ in (1.8) (with our choces of constants

in (1.10)) is analytic inside the circle $|s - u| = R + \varepsilon$, with $\frac{1}{2} < u \leq 2$ and sufficiently small $\varepsilon > 0$, and satisfies the following upper bound(1.9)

B(s) Đ b0,

with b0 = 1.0132, on the circle s u = R with ω = ξ(1/2) and Ω = 47.545/ξ(1/2).

Remark. In our forthcoming applications in [9], we will take $\omega = \xi(\frac{1}{2})$.
In [18], we have chosen that

$$\omega = \xi(1/2) > 0.497, \quad W_1 = 4R + R^{1/4} + 1/2, \quad W_2 = 4R + 1/2,$$

(1.10) $$q = \frac{(R+10)\log(R/2) + 4\log\Omega}{4\acute{\gamma}R^{1/4}},$$

$$K = \frac{\log\left(\frac{10}{27}\left(15 + \frac{4}{R^{3/4}}\right)R^{1/4} + \frac{2}{3R^{3/4}}\right) + 2\log R}{\log 2},$$

with $\acute{\gamma} := 0.3183$ and $\Omega = \frac{47.545}{\xi(1/2)}$ in the expression of q.

Lemma 2. *We have*

(1.11) $$|\xi(s)| \leq 47.545\left(\frac{R}{2}\right)^{R/4+5/2},$$

on the circle $|s - u| = R$, $\frac{1}{2} < u \leq 2$, *for* $R \geq R_0$, *with* R_0 *defined at the end of Section* 1.

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